Analysis of Numerical Dissipation and Dispersion

The exact solution of the discretized equations satisfies a PDE which is generally different from the one to be solved.

Original PDE

$$\frac{\partial u}{\partial t} + \beta u = 0$$

Modified equation

$$Au^{n+1} = Bu^n$$

$$\frac{\partial u}{\partial t} + \beta u = \sum_{p=1}^{\infty} \alpha_{2p} \frac{\partial^{2p} u}{\partial x^{2p}} + \sum_{p=1}^{\infty} \alpha_{2p+1} \frac{\partial^{2p+1} u}{\partial x^{2p+1}}$$

PDEs are difficult or impossible to solve analytically but their qualitative behaviour is easier to predict than that of discretized equations.

- Expand all nodal values in the difference scheme in a double Taylor series about a single point (x_i, t^n) of the space-time mesh to obtain a PDE.
- Express high-order time derivatives as well as mixed derivatives in terms of space derivatives using this PDE to transform it into the desired form.

Let us take pure convection equation,

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0, \quad v > 0$$

BDS in space, FE in time:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + v\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 \quad (upwind)$$

Taylor series expansions about the point (x_i, t^n)

$$u_i^{n+1} = u_i^n + \left(\frac{\partial u}{\partial t}\right)_i^n \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{(\Delta t)^2}{2} + \left(\frac{\partial^3 u}{\partial t^3}\right)_i^n \frac{(\Delta t)^3}{6} + \cdots$$

$$u_{i-1}^n = u_i^n - \left(\frac{\partial u}{\partial x}\right)_i^n \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{6} + \cdots$$

Substitution into the difference scheme yields original PDE as *O [Δt, Δx]* truncation error

$$\left(\frac{\partial u}{\partial t}\right)_i^n + v\left(\frac{\partial u}{\partial x}\right)_i^n = -\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{\Delta t}{2} - \left(\frac{\partial^3 u}{\partial t^3}\right)_i^n \frac{(\Delta t)^2}{6} + \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \frac{v\Delta x}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_i^n \frac{v(\Delta x)^2}{6} + \cdots \quad (1)$$

Replace both time derivatives in the RHS by space derivatives, differentiate above equation with respect to *t* and with respect to *x* multiply by *v*

$$\frac{\partial^2 u}{\partial t^2} + v\frac{\partial^2 u}{\partial x \partial t} = -\frac{\partial^3 u}{\partial t^3}\frac{\Delta t}{2} - \frac{\partial^4 u}{\partial t^4}\frac{(\Delta t)^2}{6} + \frac{\partial^3 u}{\partial x^2 \partial t}\frac{v\Delta x}{2} - \frac{\partial^4 u}{\partial x^3 \partial t}\frac{v(\Delta x)^2}{6} + \cdots \quad (2)$$

$$v\frac{\partial^2 u}{\partial t\partial x}+v^2\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^3 u}{\partial t^2\partial x}\frac{v\Delta t}{2}-\frac{\partial^4 u}{\partial t^3\partial x}\frac{v(\Delta t)^2}{6}+\frac{\partial^3 u}{\partial x^3}\frac{v^2\Delta x}{2}-\frac{\partial^4 u}{\partial x^4}\frac{v^2(\Delta x)^2}{6}+\cdots \qquad (3)$$

Subtract (3) from (2) and drop high-order terms

$$\frac{\partial^2 u}{\partial t^2}=v^2\frac{\partial^2 u}{\partial x^2}+\left[-\frac{\partial^3 u}{\partial t^3}+v\frac{\partial^3 u}{\partial t^2\partial x}+\mathcal{O}(\Delta t)\right]\frac{\Delta t}{2}+\left[v\frac{\partial^3 u}{\partial x^2\partial t}-v^2\frac{\partial^3 u}{\partial x^3}+\mathcal{O}(\Delta x)\right]\frac{\Delta x}{2}+\cdots \qquad (4)$$

Differentiate equation (4) with respect to *t*,

$$\frac{\partial^3 u}{\partial t^3}=v^2\frac{\partial^3 u}{\partial x^2\partial t}+\mathcal{O}[\Delta t,\Delta x] \qquad (5)$$

Differentiate equation (4) with respect to *x*,

$$\frac{\partial^3 u}{\partial t^2\partial x}=v^2\frac{\partial^3 u}{\partial x^3}+\mathcal{O}[\Delta t,\Delta x] \qquad (6)$$

Differentiate equation (3) with respect to *x*,

$$\frac{\partial^3 u}{\partial x^2\partial t}=-v\frac{\partial^3 u}{\partial x^3}+\mathcal{O}[\Delta t,\Delta x] \qquad (7)$$

Equations (5) and (7) imply that,

$$\frac{\partial^3 u}{\partial t^3}=-v^3\frac{\partial^3 u}{\partial x^3}+\mathcal{O}[\Delta t,\Delta x] \qquad (8)$$

Substitute (6) to (8) in (4)

$$\frac{\partial^2 u}{\partial t^2}=v^2\frac{\partial^2 u}{\partial x^2}+v^2(v\Delta t-\Delta x)\frac{\partial^3 u}{\partial x^3}+\mathcal{O}[\Delta t,\Delta x] \qquad (9)$$

Substitute (8) and (9) in (1) to obtain the modified equation

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}=-\frac{v^2\Delta t}{2}\left[\frac{\partial^2 u}{\partial x^2}+(v\Delta t-\Delta x)\frac{\partial^3 u}{\partial x^3}\right]+\frac{v^3(\Delta t)^2}{6}\frac{\partial^3 u}{\partial x^3}+\frac{v\Delta x}{2}\frac{\partial^2 u}{\partial x^2}-\frac{v(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}+\cdots \qquad (10)$$

which can be rewritten in terms of the Courant number $V=v\frac{\Delta t}{\Delta x}$ as follows

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}=\underbrace{\frac{v\Delta x}{2}(1-V)\frac{\partial^2 u}{\partial x^2}}_{\textit{numerical dissipation}}+\underbrace{\frac{v(\Delta x)^2}{6}(3V-2V^2-1)\frac{\partial^3 u}{\partial x^3}}_{\textit{numerical dispersion}}+\cdots \qquad (11)$$

The CFL stability condition $V\leq 1$ must be satisfied for the discrete problem to be well-posed. In the case $V>1$, the numerical diffusion coefficient $\frac{v\Delta x}{2}(1-V)$ is negative, which corresponds to a backward heat equation.

When $V<1$, it gives more stable upwind discretization for hyperbolic systems.